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ON THE CONVERGENCE OF INTERPOLATORY PRODUCT INTEGRATION RULES BASED ON GAUSS, RADAU AND LOBATTO POINTS

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ABSTRACT

Conditions are given which insure convergence for all Riemann-integrable functions of interpolatory product integration rules and their companion rules based on the Gauss, Radau or Lobatto points with respect to a generalized smooth Jacobi weight function.

1. Introduction

Interpolatory (or polynomial) product integration is concerned with numerical integration rules of the form

$$I(kf) \equiv \int_a^b k(x)f(x)dx = Q_n(f;k) + E_n(f;k) \tag{1}$$

where $k \in L_1(a,b)$,

$$Q_n(f;k) = \sum_{i=l_n}^{m_n} w_{ni}(k)f(x_{ni}), \tag{2}$$

the set of points

$$X = \{x_{ni}: i = l_n, \ldots, m_n;\ n = 1,2,\ldots;\ m_{n+1} - l_{n+1} > m_n - l_n\} \tag{3}$$

is specified in advance and the coefficients $\{w_{ni}(k): i = l_n, \ldots, m_n\}$ are chosen so that $E_n(f;k) = 0$ whenever $f \in \mathcal{P}_{m_n - l_n}$, the set of all polynomials of degree $\leqq m_n - l_n$. An application of interpolatory product integration is in the numerical solution of Fredholm integral equations of the second kind [7].

In the present work, the interval of integration will be finite and without loss of generality we shall assume it to be $U \equiv [-1,1]$. X will be given by the set of all Gauss, left Radau, right Radau or Lobatto points with respect to an

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admissible weight function on U, $\omega \in A(U)$, where $A(U)$ is the set of all nonnegative functions ω on U such that $\omega \in L_1(U)$ and $\omega > 0$ on some subinterval of U. For such a weight function, there exists a sequence of *orthonormal* polynomials

$$p_n(x;\omega) = k_n(\omega)z^n + \cdots; \qquad k_n(\omega) > 0, \quad n = 0,1,2,\cdots \tag{4}$$

such that

$$\int_{-1}^{1} \omega(x)p_n(x;\omega)p_m(x;\omega)dx = \delta_{nm}.$$

We shall denote the zeros of $p_n(x;\omega)$ by $x_{ni}(\omega)$, $i = 1,\ldots,n$.

For any function $f \in R(U)$, the set of all Riemann-integrable functions on U, we have the following sequence of integration rules:

$$\int_{-1}^{1} \omega(x)f(x)dx = Q_nf + E_nf \tag{5}$$

where

$$Q_nf = \sum_{i=1-r}^{n+s} \bar{\mu}_{ni}f(\bar{x}_{ni}), \tag{6}$$

r, $s \in \{0,1\}$, the $\bar{x}_{ni}$ are the zeros of

$$q_{n+r+s}(x) = (1+x)^r(1-x)^s p_n(x;\bar{\omega}) \tag{7}$$

with

$$\bar{\omega}(x) = (1+x)^r(1-x)^s\omega(x) \in A(U) \tag{8}$$

and are ordered as follows:

$$-1 \equiv \bar{x}_{n0} < \bar{x}_{n1} < \cdots < \bar{x}_{nn} < \bar{x}_{n,n+1} \equiv 1 \tag{9}$$

and $E_nf = 0$ whenever $f \in \mathcal{P}_{2n-1+r+s}$. The coefficients $\bar{\mu}_{ni}$ are interpolatory and are given by

$$\bar{\mu}_{ni} = (1+\bar{x}_{ni})^{-r}(1-\bar{x}_{ni})^{-s}\mu_{ni}(\bar{\omega}), \qquad i = 1,\ldots,n \tag{10}$$

where the $\mu_{ni}(\bar{\omega})$ are the Christoffel numbers defined for any $v \in A(U)$ by

$$\mu_{ni}(v) = \sum_{k=0}^{n-1} p_k(x_{ni}(v);v)^{-2}. \tag{11}$$

When $r = 1$

$$2\bar{\mu}_{n0} = \int_{-1}^{1} \omega(x)(1-x)dx - \sum_{i=1}^{n} \bar{\mu}_{ni}(1-\bar{x}_{ni})$$

and when $s = 1$

$$2\bar{\mu}_{n,n+1} = \int_{-1}^{1} \omega(x)(1+x)dx - \sum_{i=1}^{n} \bar{\mu}_{ni}(1+\bar{x}_{ni}).$$

Note that

$$\bar{x}_{ni} = x_{ni}(\bar{\omega}), \qquad i = 1,\dots,n. \tag{12}$$

$Q_n f$ is the n-point Gauss rule with respect to the weight function ω when $r = s = 0$, the $(n+1)$-point left Radau rule, when $r = 1$, $s = 0$, the $(n+1)$-point right Radau rule, when $r = 0$, $s = 1$, and the $(n+2)$-point Lobatto rule, when $r = s = 1$. By [2, p. 126] all the coefficients $\bar{\mu}_{ni}$ are positive so that $Q_n f \to I(\omega f)$ for all $f \in R(U)$ ([2, p. 130]). Consequently, by the theorem in [2, p. 131] which also holds for weighted integrals [6], $\bar{\mu}_{n0}$ and $\bar{\mu}_{n,n+1} \to 0$. Hence the *modified* integration rules

$$\hat{Q}_n f = \sum_{i=1}^{n} \bar{\mu}_{ni} f(\bar{x}_{ni}) = \sum_{i=1}^{n} \frac{\mu_{ni}(\bar{\omega}) f(x_{ni}(\bar{\omega}))}{(1+x_{ni}(\bar{\omega}))^r (1-x_{ni}(\bar{\omega}))^s} \tag{13}$$

converge to $I(\omega f)$ for all $f \in R(U)$. Note that for the Gauss case, $r = s = 0$, $\hat{Q}_n f = Q_n f$.

2. Product integration. Preliminaries and theorems

For any $\omega \in A(U)$ and any pair $r, s \in \{0,1\}$ we define the set $\bar{X}$ of Gauss, Radau or Lobatto points by

$$\bar{X} = \{\bar{x}_{ni} : i = 1-r,\dots,n+s;\ n = 1,2,\dots\}. \tag{14}$$

Then the interpolatory product integration rule determined by $\bar{X}$ is given by

$$\bar{Q}_n(f;k) = \sum_{i=1-r}^{n+s} \bar{w}_{ni}(k) f(\bar{x}_{ni}) \tag{15}$$

and is exact for all $f \in \mathcal{P}_{n-1+r+s}$. If $\bar{L}_n(f;x)$ is the Lagrange interpolation polynomial of degree $n+r+s-1$ interpolating to $f(x)$ at $\bar{x}_{ni}$, $i = 1-r,\dots,n+s$, then [2, p. 75]

$$\bar{Q}_n(f;k) = \int_{-1}^{1} k(x)\bar{L}_n(f;x)dx. \tag{16}$$

Since in our case

$$\bar{L}_n(f;x) = \sum_{i=1-r}^{n+s} \bar{l}_i(x) f(\bar{x}_{ni}) \tag{17}$$

where

$$\bar{l}_i(x) = \frac{q_{n+r+s}(x)}{(x-\bar{x}_{ni})q'_{n+r+s}(\bar{x}_{ni})} \tag{18}$$

we have that

$$\bar{w}_{ni}(k) = \int_{-1}^{1} k(x)\bar{l}_i(x)dx. \tag{19}$$

Now, for a product integration rule $Q_n(f;k)$ based on the zeros $x_{ni}(v)$ of $p_n(x;v)$ where $v \in A(U)$, it has been shown [3, 8] that

$$w_{ni}(k) = \mu_{ni}(v)S^K_{n-1}(v;x_{ni}(v)), \qquad i=1,\dots,n \tag{20}$$

where $S^K_{n-1}(v;x)$ is the nth partial sum of the Fourier series in the orthonormal polynomials $p_j(x;v)$ for the function K defined by

$$K(x) = k(x)/v(x). \tag{21}$$

Thus

$$S^K_{n-1}(v;x) = \sum_{j=0}^{n-1} b^K_j p_j(x;v) \tag{22}$$

where

$$b^K_j = \int_{-1}^{1} v(x)K(x)p_j(x;v)dx = \int_{-1}^{1} k(x)p_j(x;v)dx.$$

We now show that in our case

$$\bar{w}_{ni}(k) = \bar{\mu}_{ni}S^K_{n-1}(\bar{\omega};\bar{x}_{ni}), \qquad i=1,\dots,n \tag{23}$$

where

$$K(x) = k(x)/\omega(x) = k(x)(1+x)^r(1-x)^s/\bar{\omega}(x). \tag{24}$$

In general, for $i=1,\dots,n$

$$\begin{aligned}
\bar{w}_{ni}(k) &= \int_{-1}^{1} \frac{k(x)q_{n+r+s}(x)}{(x-\bar{x}_{ni})q'_{n+r+s}(\bar{x}_{ni})}dx \\
&= (1+\bar{x}_{ni})^{-r}(1-\bar{x}_{ni})^{-s}\int_{-1}^{1} \frac{k(x)(1+x)^r(1-x)^s p_n(x;\bar{\omega})}{(x-\bar{x}_{ni})p'_n(\bar{x}_{ni};\bar{\omega})}dx \\
&= (1+\bar{x}_{ni})^{-r}(1-\bar{x}_{ni})^{-s}w_{ni}(k(x)(1+x)^r(1-x)^s) \\
&= (1+\bar{x}_{ni})^{-r}(1-\bar{x}_{ni})^{-s}\mu_{ni}(\bar{\omega})S^K_{n-1}(\bar{\omega};x_{ni}(\bar{\omega})) \\
&= \bar{\mu}_{ni}S^K_{n-1}(\bar{\omega};\bar{x}_{ni}).
\end{aligned}$$

In this chain, we have used (19), (18), (7), (20), (24), (10) and (12).

We are now almost ready to state our theorem on the convergence of $\bar{Q}_n(f;k)$ to $I(kf)$. This theorem assumes ω to be a generalized smooth Jacobi weight function studied by Nevai [4] and Badkov [1] and generalizes Theorems 2 and 3 in [8]. The generalized smooth Jacobi weight function is defined as follows:

DEFINITION. ω is a generalized smooth weight function, $\omega \in GSJ \equiv GSJ(\alpha,\beta)$, if ω can be written in the form

$$\omega(x) = H(x)(1-x)^{\alpha}(1+x)^{\beta}\prod_{j=1}^{m}|x-t_j|^{\gamma_j} \tag{25}$$

where $-1 < t_1 < \cdots < t_m < 1$, α, β, $\gamma_j > -1$, $j = 1,\dots,m \geqq 0$, $H(x) > 0$ on U, $H \in C(U)$ and the modulus of continuity of H, $w(H,t)$, satisfies

$$\int_0^1 t^{-1}w(H,t)dt < \infty.$$

Since the theorems in [8] deal not only with the convergence of $\bar{Q}_n(f;k)$ to $I(kf)$ but also with the convergence to $I(|k|f)$ of the companion rules

$$|\bar{Q}_n|(f;k) = \sum_{i=1-r}^{n+s}|\bar{w}_{ni}(k)|f(x_{ni}), \tag{26}$$

our theorem will aso include this feature. For the practical implications of the convergence of the companion rules, see [8].

We now state our main result.

THEOREM 1. *Let $\omega \in GSJ(\alpha,\beta)$ and let $r,s \in \{0,1\}$. If for some $p > 1$, k satisfies*

$$\int_{-1}^{1}\left|k(x)(1-x)^{A}(1+x)^{B}\prod_{j=1}^{m}|x-t_j|^{C_j}\right|^{p}dx < \infty \tag{27}$$

where

$$A = -\max[(2\alpha+1-2s)/4, 0],$$

$$B = -\max[(2\beta+1-2r)/4, 0],$$

$$C_j = -\max[\gamma_j/2, 0], \qquad j = 1,\dots,m,$$

then $\bar{Q}_n(f;k) \to I(kf)$ and $|\bar{Q}_n|(f;k) \to I(|k|f)$ as $n \to \infty$ for all $f \in R(U)$.

We shall prove this theorem in Section 3. For the moment, we note that we can state an equivalent theorem by looking at the problem from another point of view. Starting with any orthonormal polynomial $p_n(x;v)$ where $v \in GSJ(\alpha,\beta)$, we can define an interpolatory product integration rule based on the zeros of $(1+x)^r(1-x)^s p_n(x;v)$. If the weight function $(1+x)^{-r}(1-x)^{-s}v(x) \in A(U)$, we are back to our previous case with $\bar{\omega}(x)=v(x)$. In this case we have the following equivalent formulation of Theorem 1.

THEOREM 1'. *Let $r,s \in \{0,1\}$ and let $v \in GSJ(\alpha,\beta)$ be such that $\alpha - s$, $\beta - r > -1$. Let $Q_n(f;k)$ be the interpolatory product integration rule based on the zeros of $(1+x)^r(1-x)^s p_n(x;v)$ and let k satisfy*

$$\int_{-1}^{1} \left| k(x)(1-x)^{A'}(1+x)^{B'} \prod_{j=1}^{m} |x-t_j|^{C'_j} \right|^p dx < \infty$$

for some $p > 1$, where

$$A' = -\max[(2\alpha+1-4s)/4, 0],$$

$$B' = -\max[(2\beta+1-4r)/4, 0],$$

$$C'_j = -\max\{\gamma_j/2, 0], \qquad j=1,\dots,m.$$

Then $Q_n(f;k) \to I(kf)$ and $|Q_n|(f;k) \to I(|k|f)$ as $n \to \infty$ for all $f \in R(U)$.

From this formulation, which follows from Theorem 1 by replacing α by $\alpha - s$ and β by $\beta - r$, we see that if we have a set of zeros of an orthonormal polynomial $p_n(x;v)$ with $v \in GSJ(\alpha,\beta)$ which we wish to use for product integration, then provided that $(1+x)^{-r}(1-x)^{-s}v(x)$ is admissible for some $r+s>0$, we can relax the condition on k needed to insure convergence by adjoining one or both endpoints to the set of zeros. Alternatively, if we have a fixed k, we can extend the range of values of α and β for which we have convergence by 2 at the upper end at the cost of reducing it by 1 at the lower end.

In case $(1+x)^{-r}(1-x)^{-s}v(x)$ is not admissible, we may still have convergence of $Q_n(f;k)$ to $I(kf)$. This was shown in [5] where we proved that if $v \in GSJ(\alpha,\beta)$, then $Q_n(f;k) \to I(kf)$ for all $f \in C(U)$ if

(1) $k \in L \log^+ L(U)$,

(2) $(1-x)^{-s+1/4}(1+x)^{-r+1/4}v^{1/2} \in L_1(U)$,

(3) $k(x)(1-x)^{s-1/4}(1+x)^{r-1/4}v^{-1/2} \in L_1(U)$.

Thus, if $r=1$ and $-\frac{1}{2} < \beta \leqq 0$ or $s=1$ and $-\frac{1}{2} < \alpha \leqq 1$, we have convergence for all $f \in C(U)$ if k satisfies conditions (1) and (3). However, the weight function $(1+x)^{-r}(1-x)^{-s}v(x)$ is not admissible. It is an open question whether we have convergence for all $f \in R(U)$ in these circumstances. Similarly, the question of the convergence of the companion rule is not settled in this case.

3. Proof of Theorem 1

Much of this proof is modelled on that of Theorem 2 in [8] to which we refer the reader for some of the details.

We shall prove the theorem for the modified rules

$$\hat{Q}_n(f;k) = \sum_{i=1}^{n} \bar{w}_{ni}(k) f(\bar{x}_{ni}) \tag{28}$$

and

$$|\hat{Q}_n|(f;k) = \sum_{i=1}^{n} |\bar{w}_{ni}(k)| f(\bar{x}_{ni}). \tag{29}$$

Then, since when $r = 1$

$$2\bar{w}_{n0}(k) = I((1-x)k) - \hat{Q}_n(1-x;k)$$

and when $s = 1$

$$2\bar{w}_{n,n+1}(k) = I((1+x)k) - \hat{Q}_n(1+x;k)$$

it follows from the convergence of $\hat{Q}_n(f;k)$ that $\bar{w}_{n0}(k)$, $\bar{w}_{n,n+1}(k) \to 0$ so that $\bar{Q}_n(f;k) \to I(kf)$ and $|\bar{Q}_n|(f;k) \to I(|k|f)$ as $n \to \infty$ for all $f \in R(U)$.

As in [8], we prove our result for the modified companion rule (29) since the result for (28) may be proved by a parallel argument, differing only in the deletion of some of the absolute value signs. Assume now that $f \in R(U)$ so that $|f|$ is bounded on U with least upper bound $M(f)$ and let $k' \in L_1(U)$. Then we have (cf. (3.1) in [8])

$$\begin{aligned} \left| |\hat{Q}_n|(f;k) - I(|k|f) \right| &\leqq M(f) \int_{-1}^{1} |k'(x) - k(x)|\, dx + M(f) \sum_{i=1}^{n} |\bar{w}_{ni}(k-k')| \\ &\quad + \left| \sum_{i=1}^{n} |\bar{w}_{ni}(k')| f(\bar{x}_{ni}) - I(|k'|f) \right| . \end{aligned} \tag{30}$$

Now, let K' be a polynomial of degree m and define

$$k'(x) = K'(x)\omega(x). \tag{31}$$

From the definition of $S^{K'}_{n-1}$, it follows that for $n > m$ we have

$$S^{K'}_{n-1}(\bar{\omega}; x) = K'(x).$$

Hence, by (23), $\bar{w}_{ni}(k') = \bar{\mu}_{ni} K'(x_{ni})$, $i = 1, \dots, n$ for all $n > m$. Inequality (30) now becomes, for $n > m$,

$$|\,|\hat{Q}_n|(f;k)-I(|k|f)|\leqq M(f)\int_{-1}^{1}|K'(x)-K(x)|\,\omega(x)dx$$

$$+M(f)\sum_{i=1}^{n}\bar{\mu}_{ni}\,|S_{n-1}^{K-K'}(\bar{\omega};\bar{x}_{ni})|$$

$$+|\hat{Q}_n(|K'|f)-I(\omega|K'|f)| \tag{32}$$

where $\hat{Q}_n$ is given by (13).

We now show that if $S\in\mathcal{P}_{n-1}$, then

$$\hat{Q}_n(|S|)\leqq C\|S\|_{1,\omega} \tag{33}$$

for some positive constant C, where, for any p, $1\leqq p<\infty$,

$$\|g\|_{p,v}=\left(\int_{-1}^{1}|v(x)g(x)|^p dx\right)^{1/p}.$$

This follows from the result in Nevai [4, Eq. 2.4], a special case of which states that for any $v(t)$ of the form $v(t)=(1-x)^a(1+x)^b$,

$$\sum_{i=1}^{n}\mu_{ni}(\bar{\omega})v(x_{ni}(\bar{\omega}))|S(x_{ni}(\bar{\omega}))|\leqq C\int_{-1}^{1}|S(t)|\,v(t)\bar{\omega}(t)dt.$$

Setting $a=-r$, $b=-s$ and using (8), (10) and (12) yields (33). Hence

$$\sum_{i=1}^{n}\bar{\mu}_{ni}\,|S_{n-1}^{K-K'}(\bar{\omega};\bar{x}_{ni})|\leqq C\|S_{n-1}^{K-K'}\|_{1,\omega} \tag{34}$$

where $S_{n-1}\equiv S_{n-1}(\bar{\omega};x)$.

If we now define $u(x)$ by

$$u(x)=(1-x)^{A+\alpha}(1+x)^{B+\beta}\prod_{j=1}^{m}|x-t_j|^{C_j+\gamma_j} \tag{35}$$

then (27) and (24) imply that $\|K\|_{p,u}<\infty$ for some $p>1$. We now choose p' so that $p>p'>1$ and so that p' be as close to 1 as we need. Then by the Hölder inequality

$$\|S_{n-1}^{K-K'}\|_{1,\omega}\leqq C_1\|S_{n-1}^{K-K'}\|_{p',u}.$$

We now use a result by Badkov [1] on the mean convergence of generalized smooth Jacobi series. This states that if

$$v(x)=(1-x)^a(1+x)^b\prod_{j=1}^{m}|x-t_j|^{c_j}$$

then sufficient conditions for the mean convergence in the L_q norm of $v(x)S_{n-1}(\bar{\omega};x)$ for all g such that $\|g\|_{q,v}<\infty$ for some $q>1$ are that

$$\left|a+\frac{1}{q}-\frac{\alpha+s+1}{2}\right|<\min\left(\frac{1}{4},\frac{\alpha+s+1}{2}\right),$$

$$\left|b+\frac{1}{q}-\frac{\beta+r+1}{2}\right|<\min\left(\frac{1}{4},\frac{\beta+r+1}{2}\right),$$

$$c_j+1/q<\min(\gamma_j+1,\gamma_j/2+1),\qquad j=1,\dots,m.$$

Now these conditions are fulfilled for $a=A+\alpha$, $b=B+\beta$ and $c_j=C_j+\gamma_j$, $j=1,\dots,m$ and $q=p'$ sufficiently close to 1. Hence, we have that

$$\|S_{n-1}^{K-K'}\|_{p',u}\leqq C_2\|K-K'\|_{p',u}\leqq C_3\|K-K'\|_{p,u}$$

so that

$$\hat{Q}_n(|S_{n-1}^{K-K'}|)\leqq C_4\|K-K'\|_{p,u}.$$

Similarly, by the Hölder inequality,

$$\int_{-1}^{1}|K'(x)-K(x)|\,\omega(x)dx\leqq C_5\|K-K'\|_{p,u}.$$

Since $\|K\|_{p,u}<\infty$, there exists $K'\in\mathcal{P}_m$ for some m such that $\|K-K'\|_{p,u}$ is sufficiently small. Finally, since $\hat{Q}_n(|K'|f)\to I(\omega\,|K'|f)$ as $n\to\infty$ for all $f\in R(U)$, we are through.

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